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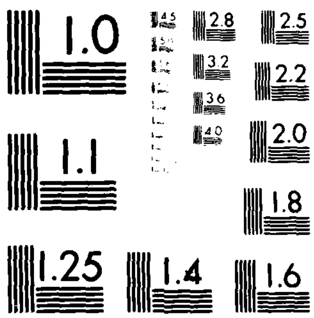

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A MOMENT SOLUTION FOR ELECTROMAGNETIC COUPLING THROUGH A SMALL APERTURE

by

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Technical Report No. 14

December 1981

Contract No. N00014-76-C-0225

Prepared for

DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
ARLINGTON, VIRGINIA 22217

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TR-81-13	2. GOVT ACCESSION NO. AD-A1	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A MOMENT SOLUTION FOR ELECTROMAGNETIC COUPLING THROUGH A SMALL APERTURE		5. TYPE OF REPORT & PERIOD COVERED Technical Report No. 14 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Joseph R. Mautz Roger F. Harrington		8. CONTRACT OR GRANT NUMBER(s) N00014-76-C-0225
9. PERFORMING ORGANIZATION NAME AND ADDRESS Dept. of Electrical & Computer Engineering Syracuse University Syracuse, New York 13210		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, Virginia 22217		12. REPORT DATE January 1982 13. NUMBER OF PAGES 52
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Aperture admittamce
Aperture in waveguide
Coupling through aperture
Moment solution
Small aperture
Waveguide coupling

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A three-term moment solution for electromagnetic coupling through a small aperture in a conducting screen is obtained. This solution gives the equivalent magnetic current in the aperture as a linear combination of the three quasi-static magnetic currents responsible for the three dipole fields seen in the far zone. Because only the dipole effects of the quasi-static currents are known, the elements of the excitation vector and the moment matrix can not be evaluated directly. Reciprocity is used to express the elements of the excitation vector in terms of the short-circuit incident fields. The

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S/N 0102-014-6601

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20. ABSTRACT (Continued)

imaginary parts of the elements of the moment matrix are evaluated by forcing the moment solution to be equal to the solution predicted by conventional small aperture theory. The real parts of the elements of the moment matrix are extracted from the quadratic expression for the power radiated by the magnetic current. The solution is applied to three examples - an aperture in a conducting plane, an aperture in a transverse wall between two rectangular waveguides, and an aperture in a lateral wall of a rectangular waveguide coupling to a half-space.

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I. INTRODUCTION

Consider electromagnetic coupling through a small aperture in an infinitely thin perfectly conducting screen separating two regions. Impressed sources in one or both regions cause electromagnetic power to flow through the aperture. The field due to the aperture is the actual field minus the short-circuit field. The short-circuit field is the field that would result if the aperture were closed with an infinitely thin sheet of perfect electric conductor. The problem is to find the far field due to the aperture.

The method of solution is similar to that used in [1], and is summarized as follows. We close the aperture with an infinitely thin sheet of perfect electric conductor and provide for the tangential electric field originally present in the aperture by attaching an unknown magnetic surface current $\underline{M}$ to one side of the sheet and $-\underline{M}$ to the other side of the sheet. Continuity of the tangential magnetic field across the sheet gives an integral equation for $\underline{M}$. This integral equation is then solved numerically by means of the method of moments. The solution is expressed in terms of generalized network parameters [2].

Specifically, $\underline{M}$ is expressed as a linear combination of the quasi-static magnetic currents $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$, which are equivalent sources whose far fields form a basis for the far field of $\underline{M}$. This linear combination is substituted into the integral equation for $\underline{M}$, and then the integral equation for $\underline{M}$ is tested with $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$ successively. The result is a system of three equations in three unknowns. In matrix notation, these equations state that the product of a square matrix with a column vector of the unknowns is equal to a column vector of known

coefficients. This matrix equation is called the moment equation. The square matrix is called the moment matrix, and the column vector of known coefficients is called the excitation vector.

The elements of the excitation vector are simply related to the short-circuit incident fields evaluated at the aperture. The imaginary parts of the elements of the moment matrix are obtained by equating the solution of the moment matrix to the solution predicted by conventional small aperture theory [3], [4]. The real parts of the elements of the moment matrix are the coefficients appearing in the quadratic expression for the power radiated by $\underline{M}$.

The far field due to the aperture is the far field due to $\underline{M}$. The solution to the moment equation gives $\underline{M}$ as a linear combination of $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$. The far field from each of $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$ is equal to the far field from a current element.

The moment solution so obtained is applied to three examples. The first example is an aperture in an infinite conducting plane, the second is an aperture in a transverse wall between two rectangular waveguides, and the third is an aperture in a lateral wall of a rectangular waveguide coupling to a half-space. In all three examples, the far field (due to the aperture) obtained by means of the moment solution is the same as that obtained by means of the radiation reaction method devised by Collin [5],[6].

In Appendix A, it is shown that the moment solution satisfies conservation of power. Conservation of power means that the power entering the aperture is equal to the power leaving the aperture. In Appendix B, it is

shown that the moment solution satisfies reciprocity in the sense that remote source and measurement points on opposite sides of the aperture can be interchanged.

II. STATEMENT OF THE PROBLEM

Consider a small aperture in an infinitely thin perfectly conducting plane screen separating two regions. Region a to the left of the aperture is filled with loss-free homogeneous material whose permeability is μ_a and whose permittivity is ε_a. Region b to the right of the aperture is filled with a loss-free homogeneous material characterized by μ_b and ε_b. As shown in Fig. 1, electric and magnetic current sources $\underline{J}^{ia}$ and $\underline{M}^{ia}$ are impressed in region a, and sources $\underline{J}^{ib}$ and $\underline{M}^{ib}$ are impressed in region b. All of these sources are remote from the aperture. In Fig. 1, the aperture is labeled A.

The problem is to find the far field due to the aperture for the situation described in the previous paragraph. By definition the field due to the aperture is the actual field minus the short-circuit field. The short-circuit field is the field that would result if the aperture were closed with a plane sheet of perfect electric conductor.

In Sections III-VI, a solution for the far-field due to the aperture is constructed. This solution is valid for the simple situation described in the second from the last paragraph. However, this solution can serve as an approximate solution for the more general situation in which the screen is not plane, inhomogeneities such as conducting walls or dielectric

obstacles are present, and power is dissipated. It can be expected that good accuracy will be obtained if the screen has only slight curvature near the aperture, no inhomogeneities occur near the aperture, and no appreciable power is dissipated near the aperture.

III. DEVELOPMENT OF THE MOMENT SOLUTION

The electromagnetic field in region a in Fig. 1 is simulated by the situation shown in Fig. 2. In Fig. 2, the original sources $\underline{J}^{ia}$ and $\underline{M}^{ia}$ exist in region a, the aperture is closed, and a sheet of magnetic current $-\underline{M}$ is placed just to the left of the closed aperture. Here,

$$\underline{M} = \underline{E} \times \underline{n} \tag{1}$$

where $\underline{E}$ is the electric field in the aperture in Fig. 1 and $\underline{n}$ is a unit vector normal to A. As shown in Fig. 2, $\underline{n}$ is directed away from region a. In Fig. 2, $(\underline{E}^{ia}, \underline{H}^{ia})$ is the electromagnetic field due to $(\underline{J}^{ia}, \underline{M}^{ia})$ radiating in the presence of the complete screen, and $(\underline{E}^{a}(-\underline{M}), \underline{H}^{a}(-\underline{M}))$ is that due to $-\underline{M}$ radiating in the presence of the complete screen. The complete screen is the screen with the aperture closed. The total field $(\underline{E}^{ia} + \underline{E}^{a}(-\underline{M}), \underline{H}^{ia} + \underline{H}^{a}(-\underline{M}))$ in region a in Fig. 2 is the field in region a in Fig. 1.

The field in region b in Fig. 1 is simulated by the situation shown in Fig. 3. In Fig. 3, the original sources $\underline{J}^{ib}$ and $\underline{M}^{ib}$ exist in region b, the aperture is closed, and the sheet of magnetic current $\underline{M}$ is placed just to the right of the closed aperture. In Fig. 3, $(\underline{E}^{ib}, \underline{H}^{ib})$ is the field due to $(\underline{J}^{ib}, \underline{M}^{ib})$ radiating in the presence of the complete screen, and $(\underline{E}^{b}(\underline{M}), \underline{H}^{b}(\underline{M}))$ is that due to $\underline{M}$ radiating in the presence of the complete

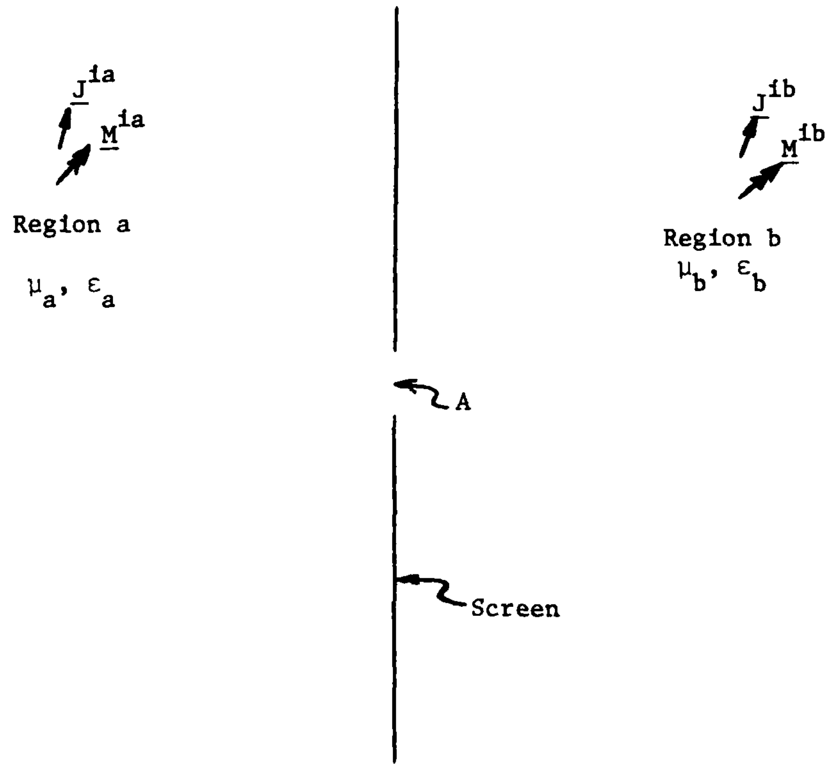

Fig. 1. Original Problem.

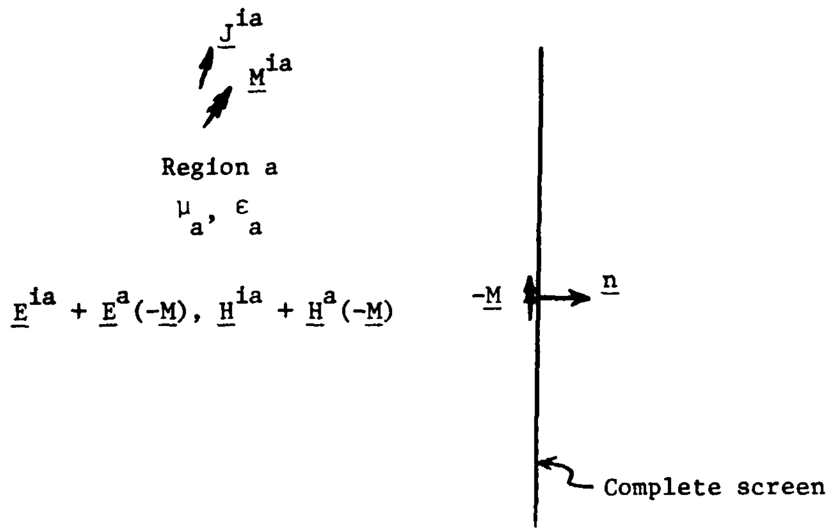

Fig. 2. Equivalence for Region a.

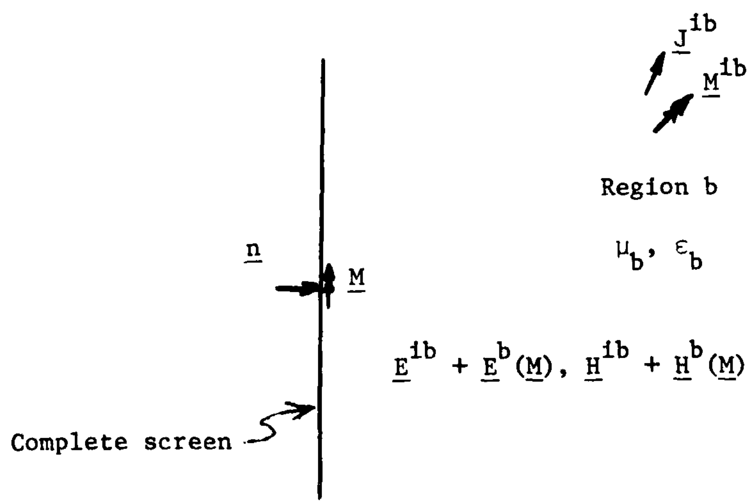

Fig. 3. Equivalence for Region b.

screen. The total field ($\underline{E}^{ib} + \underline{E}^{b}(\underline{M})$, $\underline{H}^{ib} + \underline{H}^{b}(\underline{M})$) in region b in Fig. 3 is the field in region b in Fig. 1.

The sign difference between the magnetic currents in Figs. 2 and 3 is dictated by continuity of the tangential electric field across A in Fig. 1. Continuity of the tangential magnetic field across A in Fig. 1 requires that

$$-\underline{H}^{a}_{tan}(\underline{M}) - \underline{H}^{b}_{tan}(\underline{M}) = \underline{H}^{ib}_{tan} - \underline{H}^{ia}_{tan} \quad \text{in A} \tag{2}$$

where the subscript tan denotes components tangential to A. Since the aperture is small, $\underline{M}$ may be expressed by the three term expansion

$$\underline{M} = V_1\underline{M}_1 + V_2\underline{M}_2 + V_3\underline{M}_3 \tag{3}$$

where $\underline{M}_1$ is the quasi-static current whose radiation field is that of a unit magnetic current element $K\ell = 1$ in the $\underline{t}_1$ direction, $\underline{M}_2$ is the quasi-static current whose radiation field is that of a unit magnetic current element $K\ell = 1$ in the $\underline{t}_2$ direction, and $\underline{M}_3$ is the quasi-static current whose radiation field in region a is that of an electric current element $I\ell = -j\omega\varepsilon_a$ in the $\underline{n}$ direction. The $\underline{t}_1$ and $\underline{t}_2$ directions are orthogonal directions in the plane of the aperture. All of the above three current elements are placed at the origin of coordinates. This origin is chosen to be somewhere in the aperture.

The phase of each of $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$ is taken to be constant throughout the aperture. Because their radiation fields are those of unit magnetic current elements, $\underline{M}_1$ and $\underline{M}_2$ are purely real. With regard to $\underline{M}_3$, the electric current element $I\ell = -j\omega\varepsilon_a$ in the $\underline{n}$ direction is equivalent in

region a to an infinitesimal loop of magnetic current in the plane of the aperture characterized by [7, p. 135]

$$KS = 1 \tag{4}$$

where K is the filamentary magnetic current and S is the area of the loop. It follows from (4) that $\underline{M}_3$ is also purely real.

To obtain the moment solution to (2), we substitute (3) into (2), define the symmetric product $<\underline{A}, \underline{B}>$ between two vectors $\underline{A}$ and $\underline{B}$ to be the integral of their dot product over A, and then take the symmetric product of (2) with $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$ successively. In matrix form, the result is

$$[Y^a + Y^b]\vec{V} = \vec{I}^i \tag{5}$$

where $[Y^a]$ and $[Y^b]$ are square matrices, and $\vec{V}$ and $\vec{I}$ are column vectors. The elements of $[Y^a]$, $[Y^b]$, and I^i are given by

$$\left.\begin{array}{ll} Y^a_{ij} = -<\underline{M}_i, \underline{H}^a(\underline{M}_j)> & (6) \\ Y^b_{ij} = -<\underline{M}_i, \underline{H}^b(\underline{M}_j)> & (7) \end{array}\right\} \begin{array}{l} i=1,2,3 \\ j=1,2,3 \end{array}$$

$$I^i_i = \quad <\underline{M}_i, (\underline{H}^{ib}-\underline{H}^{ia})>, i=1,2,3 \tag{8}$$

The elements of $\vec{V}$ are the coefficients V_1, V_2, and V_3 appearing in (3).

The elements (6) - (8) cannot be evaluated in a straightforward manner because the functional forms of the magnetic currents $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$ are not known. However, it has been established that

1) The magnetic currents $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$ are real.

2) The radiation field of $\underline{M}_1$ is that of the unit magnetic current element $K\ell = 1$ in the $\underline{t}_1$ direction.

3) The radiation field of $\underline{M}_2$ is that of the unit magnetic current element $K\ell = 1$ in the $\underline{t}_2$ direction.

4) In region a, the radiation field of $\underline{M}_3$ is that of the electric current element $I\ell = -j\omega\varepsilon_a$ in the $\underline{n}$ direction.

The magnetic current $\underline{M}_j$ as it appears in $\underline{H}^a(\underline{M}_j)$ in (6) is attached to the left-hand side of the complete screen. The magnetic current $\underline{M}_j$ as it appears in $\underline{H}^b(\underline{M}_j)$ in (7) is attached to the right-hand side of the complete screen. The radiation fields of $\underline{M}_1$ and $\underline{M}_2$ are those of the unit magnetic current elements mentioned in assumptions 2) and 3) regardless of the side of the complete screen to which $\underline{M}_1$ and $\underline{M}_2$ are attached. Assumption 4) states that when $\underline{M}_3$ is attached to the left-hand side of the screen, its radiation field is that of the electric current element $I\ell = -j\omega\varepsilon_a$ in the $\underline{n}$ direction. However, it follows from the result of [7, Prob. 3-6.] that when $\underline{M}_3$ is attached to the right-hand side of the complete screen, its radiation field is that of the electric current element $I\ell = -j\omega\varepsilon_b$ in the $\underline{n}$ direction.

IV. EVALUATION OF THE EXCITATION VECTOR

Assumptions 2) to 4) of Section III will now be used to evaluate the elements (8) of the excitation vector $\vec{I}^i$. In view of the definition of the symmetric product, (8) becomes

$$I_i^i = I_i^{ib} - I_i^{ia} \tag{9}$$

where

$$I_i^{ia} = \iint_A \underline{M}_i \cdot \underline{H}^{ia} ds \tag{10}$$

$$I_i^{ib} = \iint_A \underline{M}_i \cdot \underline{H}^{ib} ds \tag{11}$$

Here, ds is the differential element of area. Application of reciprocity [7, Sec. 3-8.] to the right-hand side of (10) gives

$$I_i^{ia} = \iint (-\underline{J}^{ia} \cdot \underline{E}^a(\underline{M}_i) + \underline{M}^{ia} \cdot \underline{H}^a(\underline{M}_i))ds \tag{12}$$

where $(\underline{E}^a(\underline{M}_i), \underline{H}^a(\underline{M}_i))$ is the field radiated by $\underline{M}_i$ attached to the left-hand side of the complete screen. The integration in (12) is over the domain of $\underline{J}^{ia}$ and $\underline{M}^{ia}$. Assuming that this domain is in the radiation field of $\underline{M}_i$, we may, by virtue of assumptions 2) to 4), replace (12) by

$$I_i^{ia} = \iint (-\underline{J}^{ia} \cdot \underline{E}^a(\underline{\hat{M}}_i) + \underline{M}^{ia} \cdot \underline{H}^a(\underline{\hat{M}}_i))ds \tag{13}$$

where $\underline{\hat{M}}_1$ is the unit magnetic current element $K\ell = 1$ in the $\underline{t}_1$ direction, $\underline{\hat{M}}_2$ is the unit magnetic current element $K\ell = 1$ in the $\underline{t}_2$ direction, and $\underline{\hat{M}}_3$ is the electric current element $I\ell = -j\omega\varepsilon_a$ in the $\underline{n}$ direction. We emphasize that $\underline{\hat{M}}_3$ is an electric current rather than a magnetic current. The impulsive currents $\underline{\hat{M}}_1$, $\underline{\hat{M}}_2$, and $\underline{\hat{M}}_3$ are located at the origin of coordinates. As it appears in (13), $\underline{\hat{M}}_1$ is attached to the left-hand side of the complete screen.

Application of reciprocity to the right-hand side of (13) gives

$$I_i^{ia} = \iint_A \underline{\hat{M}}_i \cdot \underline{H}^{ia} ds\ , \quad i = 1,2 \tag{14}$$

$$I_3^{ia} = -\iint_A \underline{\hat{M}}_3 \cdot \underline{E}^{ia} ds \tag{15}$$

Because of the impulsive nature of $\hat{\underline{M}}_1$, $\hat{\underline{M}}_2$, and $\hat{\underline{M}}_3$, (14) and (15) reduce to

$$
\begin{aligned}
I_1^{ia} &= H_1^{ia} \\
I_2^{ia} &= H_2^{ia} \\
I_3^{ia} &= j\omega\varepsilon_a E_3^{ia}
\end{aligned}
\tag{16}
$$

where H_1^{ia} is the $\underline{t}_1$ component of $\underline{H}^{ia}$ at the origin, H_2^{ia} is the $\underline{t}_2$ component of $\underline{H}^{ia}$ at the origin, and E_3^{ia} is the $\underline{n}$ component of $\underline{E}^{ia}$ at the origin.

A development similar to (12) - (16) reduces (11) to

$$
\begin{aligned}
I_1^{ib} &= H_1^{ib} \\
I_2^{ib} &= H_2^{ib} \\
I_3^{ib} &= j\omega\varepsilon_b E_3^{ib}
\end{aligned}
\tag{17}
$$

where H_1^{ib}, H_2^{ib}, and E_3^{ib} are, respectively, the $\underline{t}_1$ component of $\underline{H}^{ib}$, the $\underline{t}_2$ component of $\underline{H}^{ib}$, and the $\underline{n}$ component of $\underline{E}^{ib}$, all evaluated at the origin. The factor ε_b in the last of equations (17) is due to the fact that when $\underline{M}_3$ is attached to the right-hand side of the complete screen, its radiation field is that of the electric current element $I\ell = -j\omega\varepsilon_b$ in the $\underline{n}$ direction rather than $I\ell = -j\omega\varepsilon_a$ in the $\underline{n}$ direction. Substitution of (16) and (17) into (9) gives

$$
\begin{aligned}
I_1^{i} &= H_1^{ib} - H_1^{ia} \\
I_2^{i} &= H_2^{ib} - H_2^{ia} \\
I_3^{i} &= j\omega(\varepsilon_b E_3^{ib} - \varepsilon_a E_3^{ia})
\end{aligned}
\tag{18}
$$

This completes evaluation of the elements (8) of the excitation vector $\vec{I}^i$.

V. EVALUATION OF THE IMAGINARY PART OF THE MOMENT MATRIX

The imaginary parts of Y^a_{ij} of (6) and Y^b_{ij} of (7) will now be evaluated by using Bethe's theory [3]. According to the extension of this theory to allow for contrasting media [4], the far field of the magnetic current $\underline{M}$ in Fig. 2 can be approximated by the far field of the combination of the magnetic pole dipole

$$\underline{p}_m = \frac{2\mu_b}{\mu_a+\mu_b}\,[-\underline{t}_1\,\alpha_{m1}(H_1^{ia} - H_1^{ib}) - \underline{t}_2\,\alpha_{m2}(H_2^{ia} - H_2^{ib})] \tag{19}$$

and the electric charge dipole

$$\underline{p}_e = \frac{2\varepsilon_a}{\varepsilon_a+\varepsilon_b}\,\underline{n}\,\alpha_e\,(\varepsilon_a E_3^{ia} - \varepsilon_b E_3^{ib}) \tag{20}$$

Furthermore, the far field of the magnetic current $\underline{M}$ in Fig. 3 can be approximated by the far field of the combination of the magnetic pole dipole $\frac{\mu_a}{\mu_b}\underline{p}_m$ and the electric charge dipole $\frac{\varepsilon_b}{\varepsilon_a}\underline{p}_e$. In (19) and (20), α_{m1} and α_{m2} are the magnetic polarizabilities, and α_e is the electric polarizability. The polarizabilities α_{m1}, α_{m2}, and α_e are purely real and correspond, respectively, to the α_{mxx}, α_{myy}, and α_e used in [8].

Equation (19) is simpler than [4, Eq. (34)] which contains a real symmetric polarizability dyadic. However, since it is real and symmetric, this dyadic can be diagonalized by a simple rotation the coordinate axes in the plane of the aperture. Hence, (19), in which the magnetic polarizability dyadic is represented only by the diagonal elements α_{m1} and α_{m2}, can be obtained for any small aperture.

For our purposes, it is more convenient to deal with current elements, magnetic denoted by $\underline{K\ell}$ and electric denoted by $\underline{I\ell}$. These are related to $\underline{p}_m$ and $\underline{p}_e$ by

$$\underline{K\ell} = j\omega\mu_a \underline{p}_m \tag{21}$$

$$\underline{I\ell} = j\omega \underline{p}_e \tag{22}$$

Thanks to (19) and (20), (21) and (22) become

$$\underline{K\ell} = \frac{2j\omega\mu_a\mu_b}{\mu_a + \mu_b} [- \underline{t}_1 \alpha_{m1}(H_1^{ia} - H_1^{ib}) - \underline{t}_2 \alpha_{m2}(H_2^{ia} - H_2^{ib})] \tag{23}$$

$$\underline{I\ell} = \frac{2j\omega\varepsilon_a}{\varepsilon_a + \varepsilon_b} \underline{n} \alpha_e (\varepsilon_a E_3^{ia} - \varepsilon_b E_3^{ib}) \tag{24}$$

The combination of $-\underline{K\ell}$ and $-\underline{I\ell}$ replaces $-\underline{M}$ in Fig. 2.

Previously, the magnetic current $\underline{M}$ in Fig. 3 was likened to the combination of the magnetic pole dipole $\frac{\mu_a}{\mu_b} \underline{p}_m$ and the electric charge dipole $\frac{\varepsilon_b}{\varepsilon_a} \underline{p}_e$. These dipoles are equivalent to magnetic and electric current elements $\underline{K\ell'}$ and $\underline{I\ell'}$ given by

$$\underline{K\ell'} = j\omega\mu_b \left(\frac{\mu_a}{\mu_b} \underline{p}_m\right) \tag{25}$$

$$\underline{I\ell'} = j\omega \left(\frac{\varepsilon_b}{\varepsilon_a} \underline{p}_e\right) \tag{26}$$

Comparison of (25) with (21) gives

$$\underline{K\ell'} = \underline{K\ell} \tag{27}$$

Comparison of (26) with (22) gives

$$\underline{I\ell'} = \frac{\varepsilon_b}{\varepsilon_a} \underline{I\ell} \tag{28}$$

The combination of $\underline{K\ell}'$ and $\underline{I\ell}'$ replaces $\underline{M}$ in Fig. 3.

Replacement of $-\underline{M}$ in Fig. 2 by the combination of $-\underline{K\ell}$ and $-\underline{I\ell}$ and replacement of $\underline{M}$ in Fig. 3 by the combination of $\underline{K\ell}'$ and $\underline{I\ell}'$ gives the situation shown in Fig. 4. In Fig. 4, $\underline{K\ell}$ is the magnetic current element (23), and $\underline{I\ell}$ is the electric current element (24). If the aperture is sufficiently small, the electromagnetic field in Fig. 4 is a good approximation to that in Fig. 1 at points sufficiently remote from the aperture. Note that if the result of [7, Prob. 3-6.] is used to replace the electric current elements $-\underline{I\ell}$ and $\frac{\varepsilon_b}{\varepsilon_a}\underline{I\ell}$ in Fig. 4 by infinitesimal loops of magnetic current, then the magnetic current attached to the left-hand side of the screen in Fig. 4 becomes the negative of the magnetic current attached to the right-hand side of the screen in Fig. 4. This is in agreement with the fact that the magnetic current attached to the left-hand side of the screen in Fig. 2 is the negative of the magnetic current attached to the right-hand side of the screen in Fig. 3.

The combination of current elements $-\underline{K\ell}$ and $-\underline{I\ell}$ in Fig. 4 replaces $-\underline{M}$ in Fig. 2. The radiation field of the combination of $\underline{K\ell}$ and $\underline{I\ell}$ attached to the left-hand side of the complete screen is (approximately) equal to the radiation field of $\underline{M}$ attached to the left-hand side of the complete screen. The combination of $\underline{K\ell}$ and $\underline{I\ell}$ is called $\hat{\underline{M}}$ and is expressed as

$$\hat{\underline{M}} = V_1\hat{\underline{M}}_1 + V_2\hat{\underline{M}}_2 + V_3\hat{\underline{M}}_3 \tag{29}$$

where $\hat{\underline{M}}_1$, $\hat{\underline{M}}_2$, and $\hat{\underline{M}}_3$ are defined just after (13). Thanks to (23) and (24), the V's are given by

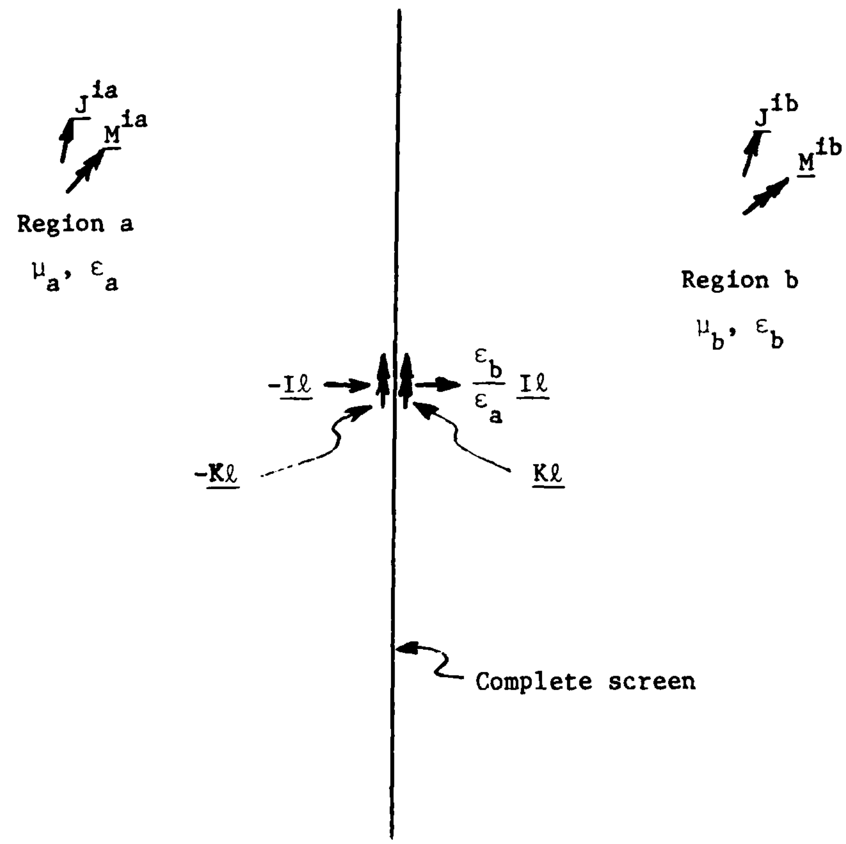

Fig. 4. Situation obtained by replacing $-\underline{M}$ in Fig. 2 and $\underline{M}$ in Fig. 3 by combinations of magnetic and electric current elements.

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} \frac{2j\omega\mu_a\mu_b\alpha_{m1}}{\mu_a + \mu_b} & 0 & 0 \\ 0 & \frac{2j\omega\mu_a\mu_b\alpha_{m2}}{\mu_a + \mu_b} & 0 \\ 0 & 0 & \frac{2\alpha_e}{j\omega(\varepsilon_a + \varepsilon_b)} \end{bmatrix} \begin{bmatrix} I_1^i \\ I_2^i \\ I_3^i \end{bmatrix} \tag{30}$$

where I_1^i, I_2^i, and I_3^i are given by (18).

Equations (29) and (30) were obtained by using Bethe's theory extended to allow for contrasting media. This theory is called conventional small aperture theory. For calculation of the far field, the moment solution (3) is equivalent to (29) where the V's are the elements of the column vector $\vec{V}$ which satisfies the moment equation (5). Hence, the moment solution will be equal to the result of conventional small aperture theory if the elements of the solution $\vec{V}$ to (5) are given by (30). By forcing the elements of the solution $\vec{V}$ to (5) to be equal to (30) for arbitrary values of I_1^i, I_2^i, and I_3^i, we obtain that $[Y^a + Y^b]^{-1}$ is equal to the square matrix in (30). This result implies that

$$[Y^a+Y^b] = \begin{bmatrix} \frac{\mu_a + \mu_b}{2j\omega\mu_a\mu_b\alpha_{m1}} & 0 & 0 \\ 0 & \frac{\mu_a + \mu_b}{2j\omega\mu_a\mu_b\alpha_{m2}} & 0 \\ 0 & 0 & \frac{j\omega(\varepsilon_a+\varepsilon_b)}{2\alpha_e} \end{bmatrix} \tag{31}$$

In order to isolate $[Y^a]$ from $[Y^b]$, we rewrite (31) as

$$[Y^a+Y^b] = \begin{bmatrix} \frac{1}{2j\alpha_{m1}}\left(\frac{1}{k_a\eta_a} + \frac{1}{k_b\eta_b}\right) & 0 & 0 \\ 0 & \frac{1}{2j\alpha_{m2}}\left(\frac{1}{k_a\eta_a} + \frac{1}{k_b\eta_b}\right) & 0 \\ 0 & 0 & \frac{j}{2\alpha_e}\left(\frac{k_a}{\eta_a} + \frac{k_b}{\eta_b}\right) \end{bmatrix} \tag{32}$$

where k_a is the wave number in region a, η_a is the intrinsic impedance in region a, k_b is the wave number in region b, and η_b is the intrinsic impedance in region b. From (6) and (7) and the nature of the magnetic field operators $\underline{H}^a$ and $\underline{H}^b$, we expect the elements of $[Y^a]$ to be proportional to $1/\eta_a$ and the elements of $[Y^b]$ to be proportional to $1/\eta_b$. It is now evident from (32) that

$$[Y^a] = \begin{bmatrix} \frac{1}{2j\omega\mu_a\alpha_{m1}} & 0 & 0 \\ 0 & \frac{1}{2j\omega\mu_a\alpha_{m2}} & 0 \\ 0 & 0 & \frac{j\omega\varepsilon_a}{2\alpha_e} \end{bmatrix} \tag{33}$$

$$[Y^b] = \begin{bmatrix} \frac{1}{2j\omega\mu_b\alpha_{m1}} & 0 & 0 \\ 0 & \frac{1}{2j\omega\mu_b\alpha_{m2}} & 0 \\ 0 & 0 & \frac{j\omega\varepsilon_b}{2\alpha_e} \end{bmatrix} \tag{34}$$

The admittance matrices (33) and (34) came out purely imaginary because no radiation terms were included in the magnetic pole dipole (19) and the electric charge dipole (20). Now, $[Y^a]$ and $[Y^b]$ have real parts which, although usually small compared to the imaginary parts, become important if the imaginary parts are cancelled by means of electromagnetic interaction.

VI. EVALUATION OF THE REAL PART OF THE MOMENT MATRIX

In this section, the real parts of Y^a and Y^b are evaluated and the far field due to the aperture is obtained from the moment solution for $\underline{M}$.

According to (6), the real part of Y^a_{ij}, denoted by $\mathrm{Re}(Y^a_{ij})$, is given by

$$\mathrm{Re}(Y^a_{ij}) = -\ \mathrm{Re}\langle \underline{M}_i,\ \underline{H}^a(\underline{M}_j)\rangle \tag{35}$$

Unfortunately, (35) is not amenable to calculation because the quasi-static magnetic currents $\underline{M}_i$ and $\underline{M}_j$ are not known.

With a view toward obtaining a workable formula for $\mathrm{Re}(Y^a_{ij})$, we consider the power P^a supplied by the magnetic current $\underline{M}$ of (3) attached to the left-hand side of the complete screen. If $\underline{M}$ is the only source in region a, then P^a is given by [7, Eqs. (1-63) and (1-64)]

$$P^a = -\ \mathrm{Re}\ \langle \underline{M}^*,\ \underline{H}^a(\underline{M})\rangle \tag{36}$$

where the asterisk denotes complex conjugate. Because $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$ are real, substitution of (3) into (36) gives

$$P^a = \mathrm{Re}(\tilde{V}^*\ Y^a\vec{V}) \tag{37}$$

where $\vec{V}$ is the column vector of the coefficients V_1, V_2, and V_3 in (3). Also, $\tilde{V}$ is the transpose of $\vec{V}$, and Y^a is the square matrix whose elements are defined by (6). Equation (37) can be rewritten as

$$P^a = \frac{1}{2}\ \tilde{V}^*\ [Y^a + [\tilde{Y}^a]^*]\vec{V} \tag{38}$$

From reciprocity, Y^a is a symmetric matrix so that (38) reduces to

$$P^a = \tilde{V}^*[\mathrm{Re}\ Y^a]\vec{V} \tag{39}$$

If the medium in region a is loss-free, then P^a is also equal to the integral of the real part of the outward component of the complex Poynting vector over a surface S such that the combination of S and the complete screen encloses the magnetic current $\underline{M}$. Hence,

$$P^a = \text{Re} \iint_S \underline{E}^a(\underline{M}) \times \underline{H}^{a*}(\underline{M}) \cdot \underline{ds} \tag{40}$$

where the magnitude of $\underline{ds}$ is the differential element of area and the direction of $\underline{ds}$ is that of the normal vector which points outward from S. If S is sufficiently remote from $\underline{M}$, then (40) becomes

$$P^a = \text{Re} \iint_S \underline{E}^a(\underline{\hat{M}}) \times \underline{H}^{a*}(\underline{\hat{M}}) \cdot \underline{ds} \tag{41}$$

where $\underline{\hat{M}}$ is given by (29). The right-hand side of (41) is the power radiated by the source $\underline{\hat{M}}$ attached to the left-hand side of the complete screen. Substitution of (29) into (41) gives

$$P^a = \tilde{V}^*[\hat{Y}^a]\vec{V} \tag{42}$$

where $\hat{Y}^a$ is a square matrix whose elements are given by

$$\hat{Y}^a_{ij} = \frac{1}{2} \iint_S [\underline{E}^a(\underline{\hat{M}}_j) \times \underline{H}^{a*}(\underline{\hat{M}}_i) + \underline{E}^{a*}(\underline{\hat{M}}_i) \times \underline{H}^a(\underline{\hat{M}}_j)] \cdot \underline{ds} \tag{43}$$

The magnetic current $\underline{M}$ was introduced in (36) as an independent source. Hence, $\vec{V}$ is arbitrary in (39) and (42). The fact that (39) is equal to (42) for all conceivable column vectors $\vec{V}$ implies that

$$\text{Re}[Y^a] = [\hat{Y}^a] \tag{44}$$

According to (44), knowledge of the power radiated by the source $\underline{\hat{M}}$ attached to the left-hand side of the complete screen determines $\text{Re}[Y^a]$. Retracing

the development (36) - (44) with $\underline{M}$ attached to the right-hand side of the complete screen, we obtain

$$\mathrm{Re}[Y^b] = [\hat{Y}^b] \tag{45}$$

where $\hat{Y}^b$ is a square matrix whose elements are given by

$$\hat{Y}^b_{ij} = \frac{1}{2} \iint_S [\underline{E}^b(\underline{\hat{M}}_j) \times \underline{H}^{b*}(\underline{\hat{M}}_i) + \underline{E}^{b*}(\underline{\hat{M}}_i) \times \underline{H}^{b*}(\underline{\hat{M}}_j)] \cdot \underline{ds} \tag{46}$$

According to (45), knowledge of the power radiated by the source $\underline{\hat{M}}$ attached to the right-hand side of the complete screen determines $\mathrm{Re}[Y^b]$.

Equations (44) and (45) call for replacement of (33) and (34) by

$$[Y^a] = \begin{bmatrix} \dfrac{1}{2j\omega\mu_a\alpha_{m1}} + \hat{Y}^a_{11} & \hat{Y}^a_{12} & \hat{Y}^a_{13} \\ \hat{Y}^a_{21} & \dfrac{1}{2j\omega\mu_a\alpha_{m2}} + \hat{Y}^a_{22} & \hat{Y}^a_{23} \\ \hat{Y}^a_{31} & \hat{Y}^a_{32} & \dfrac{j\omega\varepsilon_a}{2\alpha_e} + \hat{Y}^a_{33} \end{bmatrix} \tag{47}$$

and

$$[Y^b] = \begin{bmatrix} \dfrac{1}{2j\omega\mu_b\alpha_{m1}} + \hat{Y}^b_{11} & \hat{Y}^b_{12} & \hat{Y}^b_{13} \\ \hat{Y}^b_{21} & \dfrac{1}{2j\omega\mu_b\alpha_{m2}} + \hat{Y}^b_{22} & \hat{Y}^b_{23} \\ \hat{Y}^b_{31} & \hat{Y}^b_{32} & \dfrac{j\omega\varepsilon_b}{2\alpha_e} + \hat{Y}^b_{33} \end{bmatrix} \tag{48}$$

As stated just after (38), $[Y^a]$ is a symmetric matrix. Furthermore, $[Y^b]$ is also a symmetric matrix. For the situation in Fig. 4 where homogeneous media exists on both sides of the complete screen, $[\hat{Y}^a]$ and $[\hat{Y}^b]$ are diagonal matrices. However, the elements of $[\hat{Y}^a]$ and $[\hat{Y}^b]$

as obtained from (43) and (46) are valid for the more general situation in which the complete screen is not plane and loss-free inhomogeneities such as perfectly conducting walls or loss-free dielectric objects are present on either or both sides of the complete screen. Presumably, a slight curvature of the complete screen or loss-free inhomogeneities remote from the aperture will have little effect on the imaginary parts of $[Y^a]$ and $[Y^b]$. Hence, it is feasible to apply (47) and (48) when the complete screen curves slightly and/or loss-free inhomogeneities exist remote from the aperture. In these cases, $[\hat{Y}^a]$ and $[\hat{Y}^b]$ could have non-zero off-diagonal elements.

The matrix elements $\hat{Y}^a_{ij}$ and $\hat{Y}^b_{ij}$ in expressions (47) and (48) for the moment matrices $[Y^a]$ and $[Y^b]$ are defined by (43) and (46). In (43), $\underline{\hat{M}}_1$ is the unit magnetic current element $K\ell = 1$ in the $\underline{t}_1$ direction, $\underline{\hat{M}}_2$ is the unit magnetic current element $K\ell = 1$ in the $\underline{t}_2$ direction, and $\underline{\hat{M}}_3$ is the electric current element $I\ell = -j\omega\varepsilon_a$ in the $\underline{n}$ direction. As they appear in (43), $\underline{\hat{M}}_1$, $\underline{\hat{M}}_2$, and $\underline{\hat{M}}_3$ are attached to the left-hand side of the complete screen. In (46), $\underline{\hat{M}}_1$ is the unit magnetic current element $K\ell = 1$ in the $\underline{t}_1$ direction, $\underline{\hat{M}}_2$ is the unit magnetic current element $K\ell = 1$ in the $\underline{t}_2$ direction, and $\underline{\hat{M}}_3$ is the electric current element $I\ell = -j\omega\varepsilon_b$ in the $\underline{n}$ direction. As they appear in (46), $\underline{\hat{M}}_1$, $\underline{\hat{M}}_2$, and $\underline{\hat{M}}_3$ are attached to the right-hand side of the complete screen. The magnetic current elements $\underline{\hat{M}}_1$ and $\underline{\hat{M}}_2$ in (46) are the same as those in (43). However, the electric current element $\underline{\hat{M}}_3$ in (46) is different from that in (43) if $\varepsilon_a \neq \varepsilon_b$. The use of $\underline{\hat{M}}_3$ in (43) and (46) is based on the global definition that $\underline{\hat{M}}_3$ is the electric current element which is equivalent to the infinitesimal loop of magnetic current in the plane of the aperture characterized by (4).

Expressions (43) and (46) for the matrix elements $\hat{Y}^a_{ij}$ and $\hat{Y}^b_{ij}$ appearing in (47) and (48) were derived for loss-free media in regions a and b. However, (43) is valid so long as there is no loss inside S. Hence, it is feasible to apply (43) if there is some loss provided that the loss in the vicinity of the aperture is negligible. Similarly, (46) still applies if, in region b, loss occurs remote from the aperture.

Expression (43) for $\hat{Y}^a_{ij}$ was extracted from the power radiated by the source $\underline{\hat{M}}$ attached to the left-hand side of the complete screen. Since the medium inside S is loss-free, this power must be equal to the power supplied by $\underline{\hat{M}}$. An alternative expression for $\hat{Y}^a_{ij}$ will now be extracted from the power supplied by $\hat{M}$. Since $\underline{\hat{M}}_1$ and $\underline{\hat{M}}_2$ are purely real magnetic currents and $\underline{\hat{M}}_3$ is a purely imaginary electric current, the power supplied by $\underline{\hat{M}}$ is given by

$$P^a = - \operatorname{Re} \langle V_1^*\underline{\hat{M}}_1 + V_2^*\underline{\hat{M}}_2 \, , \; \underline{H}^a(\underline{\hat{M}})\rangle + \operatorname{Re} \langle V_3^*\underline{\hat{M}}_3 \, , \; \underline{E}^a(\underline{\hat{M}})\rangle \tag{49}$$

Substitution of (29) into (49) gives

$$P^a = \operatorname{Re}(\tilde{V}^* C^a \vec{V}) \tag{50}$$

where C^a is the square matrix whose elements are given by

$$C^a_{ij} = - \langle \underline{\hat{M}}_i , \underline{H}^a(\underline{\hat{M}}_j)\rangle \; , \; \begin{cases} i = 1,2 \\ j = 1,2,3 \end{cases} \tag{51}$$

$$C^a_{3j} = \langle \underline{\hat{M}}_3 , \underline{E}^a(\underline{\hat{M}}_j)\rangle \qquad j = 1,2,3 \tag{52}$$

Equation (50) can be rewritten as

$$P^a = \tilde{V}^* \hat{Y}^a \vec{V} \tag{53}$$

where

$$\hat{Y}^a = \frac{1}{2} \left[C^a + [\tilde{C}^a]^*\right] \tag{54}$$

The matrix $\hat{Y}^a$ defined by (54) is equal to the matrix $\hat{Y}^a$ appearing in (42) because the quadratic forms (42) and (53) are equal to each other for arbitrary $\vec{V}$.

From reciprocity, C^a is a symmetric matrix so that (54) reduces to

$$\hat{Y}^a = \text{Re } C^a \tag{55}$$

Thanks to the definition of the symmetric product and the definitions of $\underline{\hat{M}}_1$, $\underline{\hat{M}}_2$, and $\underline{\hat{M}}_3$, substitution of (51) and (52) for the elements of C^a in (55) gives

$$\hat{Y}^a = \begin{bmatrix} -\text{Re } H_1^a(\underline{\hat{M}}_1) & -\text{Re } H_1^a(\underline{\hat{M}}_2) & -\text{Re } H_1^a(\underline{\hat{M}}_3) \\ -\text{Re } H_2^a(\underline{\hat{M}}_1) & -\text{Re } H_2^a(\underline{\hat{M}}_2) & -\text{Re } H_2^a(\underline{\hat{M}}_3) \\ \omega\varepsilon_a \text{Im } E_3^a(\underline{\hat{M}}_1) & \omega\varepsilon_a \text{Im } E_3^a(\underline{\hat{M}}_2) & \omega\varepsilon_a \text{Im } E_3^a(\underline{\hat{M}}_3) \end{bmatrix} \tag{56}$$

where Im denotes imaginary part. In (56), $H_1^a(\underline{\hat{M}}_j)$, $H_2^a(\underline{\hat{M}}_j)$, and $E_3^a(\underline{\hat{M}}_j)$ are, respectively, the $\underline{t}_1$ component of $\underline{H}^a(\underline{\hat{M}}_j)$, the $\underline{t}_2$ component of $\underline{H}^a(\underline{\hat{M}}_j)$, and the $\underline{n}$ component of $\underline{E}^a(\underline{\hat{M}}_j)$. These three fields are evaluated at the origin. In (56), $\underline{\hat{M}}_3$ is the electric current element $I\ell = -j\omega\varepsilon_a$ in the $\underline{n}$ direction.

Consideration of the power supplied by $\underline{\hat{M}}$ attached to the right-hand side of the complete screen leads to

$$\hat{Y}^b = \begin{bmatrix} -\text{Re } H_1^b(\underline{\hat{M}}_1) & -\text{Re } H_1^b(\underline{\hat{M}}_2) & -\text{Re } H_1^b(\underline{\hat{M}}_3) \\ -\text{Re } H_2^b(\underline{\hat{M}}_1) & -\text{Re } H_2^b(\underline{\hat{M}}_2) & -\text{Re } H_2^b(\underline{\hat{M}}_3) \\ \omega\varepsilon_b \text{Im } E_3^b(\underline{\hat{M}}_1) & \omega\varepsilon_b \text{Im } E_3^b(\underline{\hat{M}}_2) & \omega\varepsilon_b \text{Im } E_3^b(\underline{\hat{M}}_3) \end{bmatrix} \tag{57}$$

Here, $H_1^b(\underline{\hat{M}}_j)$, $H_2^b(\underline{\hat{M}}_j)$, and $E_3^b(\underline{\hat{M}}_j)$ are, respectively, the $\underline{t}_1$ component of $\underline{H}^b(\underline{\hat{M}}_j)$, the $\underline{t}_2$ component of $\underline{H}^b(\underline{\hat{M}}_j)$, and the $\underline{n}$ component of $\underline{E}^b(\underline{\hat{M}}_j)$. These three fields are evaluated at the origin. In (57), $\underline{\hat{M}}_3$ is the electric current element $I\ell = -j\omega\varepsilon_b$ in the $\underline{n}$ direction.

Now that $\vec{I}^i$, Y^a, and Y^b have been evaluated, the moment equation (5) can be solved for $\vec{V}$. The solution to (5) is

$$\vec{V} = [Y^a + Y^b]^{-1}\vec{I}^i \tag{58}$$

where the elements of $\vec{I}^i$ are given by (18). The elements of Y^a and Y^b are given by (47) and (48) in which $\hat{Y}_{ij}^a$ is given by either (43) or (56), and $\hat{Y}_{ij}^b$ is given by either (46) or (57). The elements V_1, V_2, and V_3 of $\vec{V}$ of (58) give $\underline{M}$ according to (3).

The field due to the aperture consists of the fields radiated by $-\underline{M}$ in Fig. 2 and $\underline{M}$ in Fig. 3. If only the far field is of interest, $\underline{M}$ may be replaced by $\underline{\hat{M}}$ of (29). Hence, in region a, the far electric field due to the aperture is $\underline{E}^a(-\underline{\hat{M}})$ given by

$$\underline{E}^a(-\underline{\hat{M}}) = -\sum_{i=1}^{3} V_i\, \underline{E}^a(\underline{\hat{M}}_i) \tag{59}$$

where the V's are the elements of $\vec{V}$ of (58), and $\underline{E}^a(\underline{\hat{M}}_i)$ is the electric field due to $\underline{\hat{M}}_i$ attached to the left-hand side of the complete screen. Here, $\underline{\hat{M}}_1$ is the unit magnetic current element $K\ell = 1$ in the $\underline{t}_1$ direction, $\underline{\hat{M}}_2$ is the unit magnetic current $K\ell = 1$ in the $\underline{t}_2$ direction, and $\underline{\hat{M}}_3$ is the electric current element $I\ell = -j\omega\varepsilon_a$ in the $\underline{n}$ direction.

Similarly, in region b, the far electric field due to the aperture is $\underline{E}^b(\underline{\hat{M}})$ given by

$$\underline{E}^b(\underline{\hat{M}}) = \sum_{i=1}^{3} V_i\, \underline{E}^b(\underline{\hat{M}}_i) \tag{60}$$

where the V's are the elements of $\vec{V}$ of (58), and $\underline{E}^b(\hat{\underline{M}}_1)$ is the electric field due to $\hat{\underline{M}}_1$ attached to the right-hand side of the complete screen. Here, $\hat{\underline{M}}_1$ is the unit magnetic current element $K\ell = 1$ in the $\underline{t}_1$ direction, $\hat{\underline{M}}_2$ is the unit magnetic current element $K\ell = 1$ in the $\underline{t}_2$ direction, and $\hat{\underline{M}}_3$ is the electric current element $I\ell = -j\omega\varepsilon_b$ in the $\underline{n}$ direction.

VII. APERTURE IN AN INFINITE CONDUCTING PLANE

In this section, the moment solution for small aperture coupling is applied to a small aperture in an infinite conducting plane. Here, the situation is precisely the situation described in the first paragraph of Section II.

In expression (58) for $\vec{V}$, the elements of $\vec{I}^i$ are given by (18), Y^a is given by (47), and Y^b is given by (48). The matrix elements $\hat{Y}^a_{ij}$ appear in (47), and the matrix elements $\hat{Y}^b_{ij}$ appear in (48).

Now, $\hat{Y}^a_{ij}$ appears in the quadratic expression (42) for the power radiated by the source $\hat{\underline{M}}$ of (29) attached to the left-hand side of the complete conducting plane. Here, the adjective complete means that the aperture has been closed with a perfect conductor. In (29), $\hat{\underline{M}}_1$, $\hat{\underline{M}}_2$, and $\hat{\underline{M}}_3$ are the current elements defined in the second from the last paragraph of Section VI. In the case of the complete conducting plane, there are no cross-product terms in the quadratic expression for the power P^a radiated by $\hat{\underline{M}}$. It is evident that P^a is twice the power that would be radiated by $\hat{\underline{M}}$ in the homogeneous medium characterized by (μ_a, ε_a) everywhere. From [7, Eq. 2-116] for the power radiated by an electric current element in a homogeneous medium and from the dual expression for the power radiated by a magnetic current element in a homogeneous medium, we obtain

$$P^a = \frac{k_a^2}{3\pi\eta_a}\left(|V_1|^2 + |V_2|^2 + k_a^2|V_3|^2\right) \tag{61}$$

Comparison of (61) with (42) gives

$$\hat{Y}^a = \begin{bmatrix} \frac{k_a^2}{3\pi\eta_a} & 0 & 0 \\ 0 & \frac{k_a^2}{3\pi\eta_a} & 0 \\ 0 & 0 & \frac{k_a^4}{3\pi\eta_a} \end{bmatrix} \tag{62}$$

Similarly, we obtain

$$\hat{Y}^b = \begin{bmatrix} \frac{k_b^2}{3\pi\eta_b} & 0 & 0 \\ 0 & \frac{k_b^2}{3\pi\eta_b} & 0 \\ 0 & 0 & \frac{k_b^4}{3\pi\eta_b} \end{bmatrix} \tag{63}$$

In view of (18), (47), (48), (62), and (63), expression (58) for $\vec{V}$ becomes

$$V_1 = \frac{H_1^{ib} - H_1^{ia}}{\frac{k_a^2\eta_b + k_b^2\eta_a}{3\pi\eta_a\eta_b} - j\,\frac{\mu_a + \mu_b}{2\alpha_{m1}\omega\mu_a\mu_b}} \tag{64a}$$

$$V_2 = \frac{H_2^{ib} - H_2^{ia}}{\frac{k_a^2\eta_b + k_b^2\eta_a}{3\pi\eta_a\eta_b} - j\,\frac{\mu_a + \mu_b}{2\alpha_{m2}\omega\mu_a\mu_b}} \tag{64b}$$

$$V_3 = \frac{\varepsilon_b E_3^{ib} - \varepsilon_a E_3^{ia}}{\frac{\varepsilon_a + \varepsilon_b}{2\alpha_e} - j\,\frac{k_a^4\eta_b + k_b^4\eta_a}{3\pi\omega\eta_a\eta_b}} \tag{64c}$$

In region a, the far electric field due to the aperture is given by (59) in which the V's are given by (64) and $\underline{E}^a(\hat{\underline{M}}_i)$ is the far electric field due to $\hat{\underline{M}}_i$ attached to the left-hand side of the complete conducting plane. Here, $\hat{\underline{M}}_i$, i=1,2,3, are the current elements defined in the second from the last paragraph in Section VI.

It is apparent from [7, Sec. 3-13] that the far electric field $\underline{E}^a(\underline{I\ell})$ due to an electric current element $\underline{I\ell}$ attached to the left-hand side of the complete conducting plane is given by

$$\underline{E}^a(\underline{I\ell}) = -\frac{j\omega\mu_a e^{-jk_a r}}{2\pi r} \underline{I\ell}_{tan} \tag{65}$$

where r is the distance from the origin and $\underline{I\ell}_{tan}$ is the part of $\underline{I\ell}$ perpendicular to the radius vector from the origin. Similarly, the far magnetic field $\underline{H}^a(\underline{K\ell})$ due to a magnetic current element $\underline{K\ell}$ attached to the left-hand side of the complete conducting plane is given by

$$\underline{H}^a(\underline{K\ell}) = -\frac{j\omega\varepsilon_a e^{-jk_a r}}{2\pi r} \underline{K\ell}_{tan} \tag{66}$$

where $\underline{K\ell}_{tan}$ is the part of $\underline{K\ell}$ perpendicular to the radius vector from the origin. In the far zone, the electric and magnetic fields $\underline{E}^a$ and $\underline{H}^a$ due to any source in the vicinity of the origin satisfy

$$E^a = \eta_a \underline{H}^a \times \underline{u}_r \tag{67}$$

where $\underline{u}_r$ is the unit vector in the direction of the radius vector from the origin.

Expressions (65) - (67) will now be used to obtain $\underline{E}^a(\hat{\underline{M}}_i)$, i = 1,2,3. For convenience, an xyz coordinate system is constructed such that the unit vectors in the x,y, and z directions are $\underline{t}_1$, $\underline{t}_2$, and $\underline{n}$, respectively. With

this coordinate system, (65) gives

$$\underline{E}^a(\hat{\underline{M}}_3) = \underline{u}_\theta \frac{k_a^2 e^{-jk_a r} \sin\theta}{2\pi r} \tag{68}$$

where θ is the angle measured from the positive z axis and $\underline{u}_\theta$ is the unit vector in the θ direction. Similarly, (66) gives

$$\underline{H}^a(\hat{\underline{M}}_1) = -(\underline{u}_\theta \cos\theta \cos\phi - \underline{u}_\phi \sin\phi) \frac{jk_a e^{-jk_a r}}{2\pi\eta_a r} \tag{69}$$

and

$$\underline{H}^a(\hat{\underline{M}}_2) = -(\underline{u}_\theta \cos\theta \sin\phi + \underline{u}_\phi \cos\phi) \frac{jk_a e^{-jk_a r}}{2\pi\eta_a r} \tag{70}$$

where ϕ is the angle measured in the xy plane from the positive x axis and $\underline{u}_\phi$ is the unit vector in the ϕ direction. Application of (67) to (69) and (70) gives

$$\underline{E}^a(\hat{\underline{M}}_1) = (\underline{u}_\theta \sin\phi + \underline{u}_\phi \cos\theta \cos\phi) \frac{jk_a e^{-jk_a r}}{2\pi r} \tag{71}$$

and

$$\underline{E}^a(\hat{\underline{M}}_2) = (-\underline{u}_\theta \cos\phi + \underline{u}_\phi \cos\theta \sin\phi) \frac{jk_a e^{-jk_a r}}{2\pi r} \tag{72}$$

The far field $\underline{E}^b(\hat{\underline{M}}_i)$ due to $\hat{\underline{M}}_i$ attached to the right-hand side of the complete conducting plane is given for i = 1,2, and 3 by equations similar to (71), (72), and (68).

$$\underline{E}^b(\hat{\underline{M}}_1) = (\underline{u}_\theta \sin\phi + \underline{u}_\phi \cos\theta \cos\phi) \frac{jk_b e^{-jk_b r}}{2\pi r} \tag{73}$$

$$\underline{E}^b(\hat{\underline{M}}_2) = (-\underline{u}_\theta \cos\phi + \underline{u}_\phi \cos\theta \sin\phi) \frac{jk_b e^{-jk_b r}}{2\pi r} \tag{74}$$

$$\underline{E}^b(\hat{\underline{M}}_3) = \underline{u}_\theta \frac{k_b^2 e^{-jk_b r}}{2\pi r} \sin\theta \tag{75}$$

The far electric field $\underline{E}^a(-\hat{\underline{M}})$ in region a due to the aperture is given by (59) which is

$$\underline{E}^a(-\underline{\hat{M}}) = -\sum_{i=1}^{3} V_i\, \underline{E}^a(\underline{\hat{M}}_i) \tag{76}$$

where the V's are given by (64) and $\underline{E}^a(\underline{\hat{M}}_i)$, i = 1,2,3, is given by (71), (72), and (68). The far electric field $\underline{E}^b(\underline{\hat{M}})$ in region b due to the aperture is given by (60) which is

$$\underline{E}^b(\underline{\hat{M}}) = \sum_{i=1}^{3} V_i\, \underline{E}^b(\underline{\hat{M}}_i) \tag{77}$$

where the V's are given by (64) and $\underline{E}^b(\underline{\hat{M}}_i)$, i=1,2,3, is given by (73), (74), and (75).

VIII. APERTURE IN A TRANSVERSE WALL BETWEEN TWO WAVEGUIDES

In this section, the moment solution is applied to two rectangular waveguides of dimensions a by b and a' by b' separated by a transverse wall at z = 0 which contains a small aperture as shown in Fig. 5. The aperture is located at the center (x = a/2, y = b/2) of the cross section of the left-hand waveguide or, equivalently, at the center (x' = a'/2, y' = b'/2) of the cross section of the right-hand waveguide. Here, x and y are rectangular coordinates in the cross section of the left-hand waveguide, and x' and y' are rectangular coordinates in the cross section of the right-hand waveguide. The aperture is symmetric about the lines x = a/2 and y = b/2 so that the $\underline{t}_1$, $\underline{t}_2$, and $\underline{n}$ directions introduced in Section III can be taken as the x,y, and z directions, respectively. The left-hand waveguide is filled with the loss-free homogeneous medium (μ_a, ϵ_a) and the right-hand waveguide is filled with the loss-free

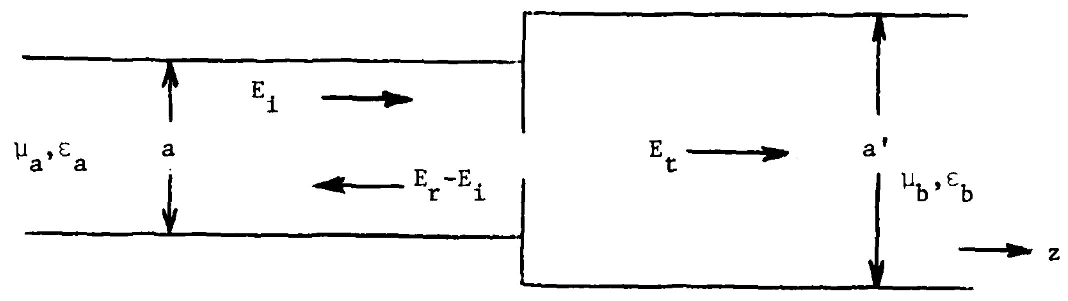

Fig. 5. Two rectangular waveguides coupled by a small aperture in a transverse wall between them.

homogeneous medium (μ_b, ε_b). It is assumed that a > b and that a' > b' so that the TE_{10} mode is the dominant mode in each waveguide. Furthermore, it is assumed that the frequency is such that only the TE_{10} mode propagates in each waveguide.

The excitation of the coupled waveguides is a TE_{10} wave incident in the left-hand waveguide. If the transverse wall were complete (no aperture), then in the right-hand waveguide there would be no field and in the left-hand waveguide the electric and magnetic fields would be $\underline{E}^{ia}$ and $\underline{H}^{ia}$ given by

$$\underline{E}^{ia} = \underline{u}_y E_i \left(e^{-j\beta z} - e^{j\beta z}\right) \sin \frac{\pi x}{a} \tag{78}$$

$$\underline{H}^{ia} = -\underline{u}_x Y E_i \left(e^{-j\beta z} + e^{j\beta z}\right) \sin \frac{\pi x}{a} + \underline{u}_z \frac{j\pi Y}{\beta a} E_i \left(e^{-j\beta z} - e^{j\beta z}\right) \cos \frac{\pi x}{a} \tag{79}$$

Here, $\underline{u}_x$, $\underline{u}_y$, and $\underline{u}_z$ are the unit vectors in the x,y, and z directions, respectively. Also, E_i is the amplitude of the incident wave, and

$$\beta = \sqrt{k_a^2 - \left(\frac{\pi}{a}\right)^2} \tag{80}$$

$$Y = \frac{\beta}{k_a \eta_a} \tag{81}$$

where

$$k_a = \omega \sqrt{\mu_a \varepsilon_a} \tag{82}$$

$$\eta_a = \sqrt{\frac{\mu_a}{\varepsilon_a}} \tag{83}$$

The superscript "a" in (78) and (79) indicates that the left-hand

waveguide is taken to be region a which appears in the general theory. The right-hand waveguide will be taken as region b of the general theory.

In the far zone ($z << 0$), the electric and magnetic fields $\underline{E}^a(\hat{\underline{M}})$ and $\underline{H}^a(\hat{\underline{M}})$ due to $\hat{\underline{M}}$ attached to the left-hand side of the complete transverse wall can be expressed as

$$\underline{E}^a(\hat{\underline{M}}) = - \underline{u}_y \, E_r \, e^{j\beta z} \sin \frac{\pi x}{a} \tag{84}$$

$$\underline{H}^a(\hat{\underline{M}}) = - \underline{u}_x \, Y \, E_r \, e^{j\beta z} \sin \frac{\pi x}{a} - \underline{u}_z \frac{j\pi Y}{\beta a} E_r \, e^{j\beta z} \cos \frac{\pi x}{a} \tag{85}$$

where E_r is an unknown coefficient. Application of reciprocity to the fields (84) - (85) and (78) - (79) gives

$$\iint_{S_{ab}} \underline{E}^a(\hat{\underline{M}}) \cdot \underline{J} \, ds = - \underline{H}^{ia} \cdot (V_1\hat{\underline{M}}_1 + V_2\hat{\underline{M}}_2) + \underline{E}^{ia} \cdot V_3\hat{\underline{M}}_3 \tag{86}$$

where S_{ab} is the cross section of the left-hand waveguide and $\underline{J}$ is an electric current sheet which, when radiating in the left-hand waveguide closed with the complete transverse wall, produces the electric and magnetic fields $\underline{E}^{ia}$ and $\underline{H}^{ia}$.

The interested reader can show that the current sheet

$$\underline{J} = - \underline{u}_y \, 2YE_i \, e^{j\beta d} \sin \frac{\pi x}{a} \tag{87}$$

placed at $z = -d$ in the cross section of the left-hand waveguide produces $\underline{E}^{ia}$ and $\underline{H}^{ia}$ for $-d < z \leq 0$. In view of the definitions of $\hat{\underline{M}}_1$, $\hat{\underline{M}}_2$, and $\hat{\underline{M}}_3$, substitution of (84) for $\underline{E}^a(\hat{\underline{M}})$, (87) for $\underline{J}$, (78) for $\underline{E}^{ia}$, and (79) for $\underline{H}^{ia}$ in (86) gives

$$E_r = \frac{2V_1}{ab} \tag{88}$$

The power P^a radiated by $\hat{\underline{M}}$ attached to the left-hand side of the complete transverse wall is given by

$$P^a = - \operatorname{Re} \iint_{S_{ab}} \underline{E}^a(\hat{\underline{M}}) \times \underline{H}^{a*}(\hat{\underline{M}}) \cdot \underline{u}_z ds \tag{89}$$

In view of (88), substitution of (84) and (85) into (89) gives

$$P^a = \frac{2Y}{ab} |V_1|^2 \tag{90}$$

A development similar to (84) - (88) shows that in the far zone ($z >> 0$), the electric and magnetic fields $\underline{E}^b(\hat{\underline{M}})$ and $\underline{H}^b(\hat{\underline{M}})$ due to $\hat{\underline{M}}$ attached to the right-hand side of the complete transverse wall are given by

$$\underline{E}^b(\hat{\underline{M}}) = \underline{u}_y E_t e^{-j\beta' z} \sin \frac{\pi x'}{a'} \tag{91}$$

$$\underline{H}^b(\hat{\underline{M}}) = - \underline{u}_x Y' E_t e^{-j\beta' z} \sin \frac{\pi x'}{a'} + \underline{u}_z \frac{j\pi Y'}{\beta' a'} E_t e^{-j\beta' z} \cos \frac{\pi x'}{a'} \tag{92}$$

where

$$\beta' = \sqrt{k_b^2 - (\frac{\pi}{a'})^2} \tag{93}$$

$$Y' = \frac{\beta'}{k_b \eta_b} \tag{94}$$

$$k_b = \omega \sqrt{\mu_b \varepsilon_b} \tag{95}$$

$$\eta_b = \sqrt{\frac{\mu_b}{\varepsilon_b}} \tag{96}$$

$$E_t = \frac{2V_1}{a'b'} \tag{97}$$

A development similar to (89) - (90) shows that the power P^b radiated by $\hat{\underline{M}}$ attached to the right-hand side of the complete transverse wall is given by

$$P^b = \frac{2Y'}{a'b'} |V_1|^2 \tag{98}$$

Comparison of (90) with (42) shows that the only non-zero element of $\hat{Y}^a$ is $\hat{Y}^a_{11}$ given by

$$\hat{Y}^a_{11} = \frac{2Y}{ab} \tag{99}$$

According to (98), the only non-zero element of $\hat{Y}^b$ is $\hat{Y}^b_{11}$ given by

$$\hat{Y}^b_{11} = \frac{2Y'}{a'b'} \tag{100}$$

It is evident from (78), (79), and (18) that the only non-zero element of the excitation vector is I^i_1 given by

$$I^i_1 = 2YE_i \tag{101}$$

In view of (47), (48), (99), (100), and (101), the only non-zero element of $\vec{V}$ of (58) is V_1 given by

$$V_1 = \frac{YE_i}{\frac{Y}{ab} + \frac{Y'}{a'b'} - j\,\frac{\mu_a + \mu_b}{4\omega\alpha_{m1}\mu_a\mu_b}} \tag{102}$$

In the far zone ($z << 0$), the electric and magnetic fields due to the aperture are the fields due to $-\hat{\underline{M}}$ attached to the left-hand side of the complete transverse wall. Thanks to (84) and (85), these fields are given by

$$\underline{E}^a(-\hat{\underline{M}}) = \underline{u}_y\, E_r\, e^{j\beta z} \sin\frac{\pi x}{a} \tag{103}$$

$$\underline{H}^a(-\hat{\underline{M}}) = \underline{u}_x YE_r e^{j\beta z} \sin\frac{\pi x}{a} + \underline{u}_z \frac{j\pi Y}{\beta a} E_r\, e^{j\beta z} \cos\frac{\pi x}{a} \tag{104}$$

where, from (88) and (102),

$$E_r = \frac{2YE_i}{ab\ (\frac{Y}{ab} + \frac{Y'}{a'b'} - j\ \frac{\mu_a + \mu_b}{4\omega\alpha_{m1}\mu_a\mu_b})} \tag{105}$$

In the far zone (z >> 0), the electric and magnetic fields due to the aperture are the fields due to $\hat{\underline{M}}$ attached to the right-hand side of the complete transverse wall. As given by (91) and (92), these fields are

$$\underline{E}^b(\hat{\underline{M}}) = \underline{u}_y\ E_t\ e^{-j\beta' z}\ \sin\frac{\pi x'}{a'} \tag{106}$$

$$\underline{H}^b(\hat{\underline{M}}) = -\ \underline{u}_x\ Y'E_t\ e^{-j\beta' z}\ \sin\frac{\pi x'}{a'} +$$

$$\underline{u}_z\ \frac{j\pi Y'}{\beta' a'}\ E_t\ e^{-j\beta' z}\ \cos\frac{\pi x'}{a'} \tag{107}$$

where, from (97) and (102),

$$E_t = \frac{2YE_i}{a'b'\ (\frac{Y}{ab} + \frac{Y'}{a'b'} - j\ \frac{\mu_a + \mu_b}{4\omega\alpha_{m1}\mu_a\mu_b})} \tag{108}$$

IX. APERTURE IN A LATERAL WALL OF A WAVEGUIDE

In this section, the moment solution is applied to a rectangular waveguide of dimensions a by b coupled to a half-space by means of a small aperture in one of its walls. As shown in Fig. 6, the aperture is located at (z = 0, x = x') in the upper (y = b) wall of the waveguide. Except for the aperture, the whole y = b plane is perfectly conducting. The aperture is symmetric about the lines z = 0 and x = x' so that the $\underline{t}_1$, $\underline{t}_2$, and $\underline{n}$ directions introduced in Section III can be taken as the z, x, and y

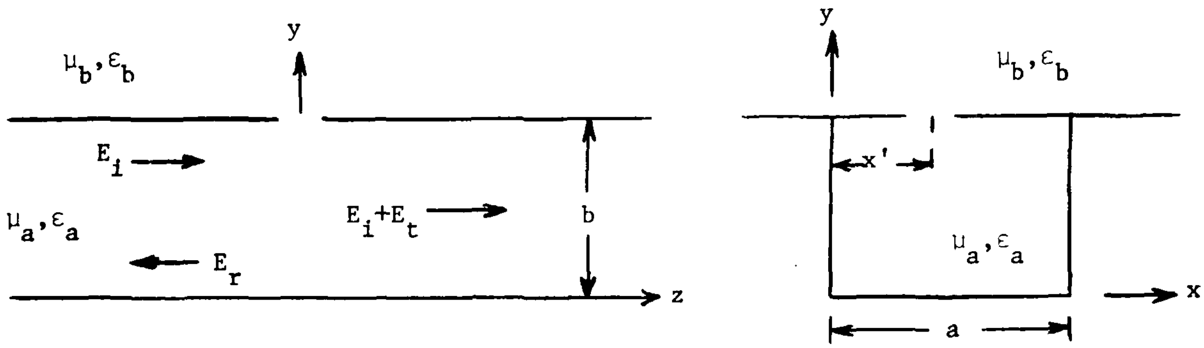

Fig. 6. A rectangular waveguide coupled to a half-space by means of a small flanged aperture.

directions, respectively. The waveguide is filled with the loss-free homogeneous medium (μ_a, ε_a) and the half-space is filled with the loss-free homogeneous medium (μ_b, ε_b). It is assumed that a > b so that the TE_{10} mode is the dominant mode in the waveguide. Furthermore, it is assumed that the frequency is such that only the TE_{10} mode propagates in the waveguide.

The excitation is a TE_{10} wave which travels from left to right in the waveguide. If the aperture were closed with a perfect conductor, then in the half-space there would be no field and in the waveguide the electric and magnetic fields would be $\underline{E}^{ia}$ and $\underline{H}^{ia}$ given by

$$\underline{E}^{ia} = \underline{u}_y E_i e^{-j\beta z} \sin \frac{\pi x}{a} \tag{109}$$

$$\underline{H}^{ia} = - \underline{u}_x YE_i e^{-j\beta z} \sin \frac{\pi x}{a} + \underline{u}_z \frac{j\pi Y}{\beta a} E_i e^{-j\beta z} \cos \frac{\pi x}{a} \tag{110}$$

Here, E_i is the wave amplitude, and β and Y are given by (80) and (81). The waveguide region is taken to be region a of the general theory. The half-space region will be taken as region b of the general theory.

Far from the aperture, the electric and magnetic fields $\underline{E}^a(-\underline{\hat{M}})$ and $\underline{H}^a(-\underline{\hat{M}})$ due to $-\underline{\hat{M}}$ attached to the inside of the upper wall of the complete waveguide at (z = 0, x = x') can be expressed as

$$\left.\begin{aligned} \underline{E}^a(-\underline{\hat{M}}) &= \underline{u}_y E_r e^{j\beta z} \sin \frac{\pi x}{a} \\ \underline{H}^a(-\underline{\hat{M}}) &= \underline{u}_x YE_r e^{j\beta z} \sin \frac{\pi x}{a} + \underline{u}_z \frac{j\pi Y}{\beta a} E_r e^{j\beta z} \cos \frac{\pi x}{a} \end{aligned}\right\} \quad z<<0 \tag{111}$$

$$\left.\begin{aligned} \underline{E}^a(-\underline{\hat{M}}) &= \underline{u}_y E_t e^{-j\beta z} \sin \frac{\pi x}{a} \\ \underline{H}^a(-\underline{\hat{M}}) &= - \underline{u}_x YE_t e^{-j\beta z} \sin \frac{\pi x}{a} + \underline{u}_z \frac{j\pi Y}{\beta a} E_t e^{-j\beta z} \cos \frac{\pi x}{a} \end{aligned}\right\} \quad z>>0 \tag{112}$$

The complete waveguide is the waveguide with the aperture closed with a perfect conductor. In (111) and (112), E_r and E_t are unknown coefficients. Application of reciprocity to the fields (111) and (109) - (110) gives

$$\iint_{S_{ab}} \underline{E}^a(-\hat{\underline{M}}) \cdot \underline{J}\, ds = \underline{H}^{ia} \cdot (V_1\hat{\underline{M}}_1 + V_2\hat{\underline{M}}_2) - \underline{E}^{ia} \cdot V_3\hat{\underline{M}}_3 \tag{113}$$

where S_{ab} is the cross section of the waveguide and $\underline{J}$ is the electric current which, when radiating inside the complete waveguide, produces the electric and magnetic fields $\underline{E}^{ia}$ and $\underline{H}^{ia}$.

It can be verified that the current sheet

$$\underline{J} = -\underline{u}_y\, 2YE_i\, e^{j\beta d} \sin\frac{\pi x}{a} \tag{114}$$

placed at $z = -d$ in the cross section of the complete waveguide produces $\underline{E}^{ia}$ and $\underline{H}^{ia}$ for $z > -d$. In view of the definitions of $\hat{\underline{M}}_1$, $\hat{\underline{M}}_2$, and $\hat{\underline{M}}_3$, substitution of (111) for $\underline{E}^a(-\hat{\underline{M}})$, (114) for $\underline{J}$, (109) for $\underline{E}^{ia}$, and (110) for $\underline{H}^{ia}$ in (113) gives

$$E_r = \frac{1}{ab}\left[\frac{-j\pi C}{\beta a} V_1 + SV_2 - \frac{jk_a^2 S}{\beta} V_3\right] \tag{115}$$

where

$$S = \sin\frac{\pi x'}{a} \tag{116}$$

$$C = \cos\frac{\pi x'}{a} \tag{117}$$

A development similar to (113) - (115) shows that

$$E_t = \frac{1}{ab}\left[\frac{-j\pi C}{\beta a} V_1 - SV_2 - \frac{jk_a^2 S}{\beta} V_3\right] \tag{118}$$

The power P^a radiated by $\hat{\underline{M}}$ attached to inside of the upper wall of the complete waveguide is, as calculated from (111) and (112),

$$P^a = \frac{abY}{2} \left[|E_r|^2 + |E_t|^2 \right] \tag{119}$$

Substitution of (115) and (118) into (119) gives

$$P^a = \tilde{V}^* \hat{Y}^a \vec{V} \tag{120}$$

where

$$\hat{Y}^a = \frac{Y}{ab} \begin{bmatrix} \dfrac{\pi^2 C^2}{\beta^2 a^2} & 0 & \dfrac{\pi k_a^2 \, SC}{\beta^2 a} \\ 0 & S^2 & 0 \\ \dfrac{\pi k_a^2 \, SC}{\beta^2 a} & 0 & \dfrac{k_a^4 \, S^2}{\beta^2} \end{bmatrix} \tag{121}$$

The half-space matrix $\hat{Y}^b$ is given by (63).

In view of (47), (48), (121), and (63), the moment matrix $[Y^a + Y^a]$ reduces to

$$[Y^a + Y^b] = \begin{bmatrix} c_1 & 0 & c_4 \\ 0 & c_2 & 0 \\ c_4 & 0 & c_3 \end{bmatrix} \tag{122}$$

where

$$c_1 = \frac{\pi^2 C^2}{k_a \eta_a \beta a^3 b} + \frac{k_b^2}{3\pi\eta_b} - \frac{j}{k_a \eta_a \alpha_{m1}} \left(\frac{\mu_a + \mu_b}{2\mu_b}\right) \tag{123a}$$

$$c_2 = \frac{\beta S^2}{k_a \eta_a ab} + \frac{k_b^2}{3\pi\eta_b} - \frac{j}{k_a \eta_a \alpha_{m2}} \left(\frac{\mu_a + \mu_b}{2\mu_b}\right) \tag{123b}$$

$$c_3 = \frac{k_a^3 \, S^2}{\eta_a \beta ab} + \frac{k_b^4}{3\pi\eta_b} + \frac{jk_b}{\eta_b \alpha_e} \left(\frac{\epsilon_a + \epsilon_b}{2\epsilon_b}\right) \tag{123c}$$

$$c_4 = \frac{\pi k_a \, SC}{\eta_a \beta a^2 b} \tag{123d}$$

From (122), we obtain

$$[Y^a+Y^b]^{-1} = \begin{bmatrix} \dfrac{c_3}{c_1c_3 - c_4^2} & 0 & \dfrac{-c_4}{c_1c_3 - c_4^2} \\ 0 & \dfrac{1}{c_2} & 0 \\ \dfrac{-c_4}{c_1c_3 - c_4^2} & 0 & \dfrac{c_1}{c_1c_3 - c_4^2} \end{bmatrix} \tag{124}$$

In view of (109) and (110), the excitation vector $\vec{I}^i$ whose elements are given by (18) reduces to

$$\vec{I}^i = \begin{bmatrix} \dfrac{-j\pi E_i C}{k_a \eta_a a} \\ \dfrac{\beta E_i S}{k_a \eta_a} \\ \dfrac{-jk_a E_i S}{\eta_a} \end{bmatrix} \tag{125}$$

Thanks to (123), substitution of (124) and (125) into (58) gives

$$V_1 = \frac{2\pi k_b CE_i \alpha_{m1}}{k_a a\Delta} [2 (\frac{\varepsilon_a + \varepsilon_b}{2\varepsilon_b}) - j \frac{2k_b^3}{3\pi} \alpha_e] \tag{126a}$$

$$V_2 = \frac{2j\beta E_i S\alpha_{m2}}{2(\frac{\mu_a + \mu_b}{2\mu_b}) + 2j (\frac{\beta S^2}{ab} + \frac{k_b^2 k_a \eta_a}{3\pi\eta_b})\alpha_{m2}} \tag{126b}$$

$$V_3 = - \frac{2\eta_b E_i S\alpha_e}{\eta_a \Delta} [2 (\frac{\mu_a + \mu_b}{2\mu_b}) + \frac{2jk_b^2 k_a \eta_a}{3\pi\eta_b} \alpha_{m1}] \tag{126c}$$

where

$$\Delta = 4\,\frac{k_b}{k_a}\left(\frac{\mu_a+\mu_b}{2\mu_b}\right)\left(\frac{\varepsilon_a+\varepsilon_b}{2\varepsilon_b}\right) + 4j\left(\frac{\varepsilon_a+\varepsilon_b}{2\varepsilon_b}\right)\left(\frac{\pi^2 C^2 k_b}{k_a \beta a^3 b} + \frac{k_b^3 \eta_a}{3\pi\eta_b}\right)\alpha_{m1} -$$

$$4j\left(\frac{\mu_a+\mu_b}{2\mu_b}\right)\left(\frac{k_a^2 S^2 \eta_b}{\beta ab\ \eta_a} + \frac{k_b^4}{3\pi k_a}\right)\alpha_e + 4\alpha_e\alpha_{m1}\left(\frac{k_b^3}{3\pi}\right)\left[\frac{k_b^3\eta_a}{3\pi\eta_b} + \frac{k_a^3 S^2}{k_b\beta ab} + \frac{\pi^2 k_b C^2}{k_a\beta a^3 b}\right] \tag{127}$$

With the xyz coordinate system in Fig. 6, the definitions of $\hat{\underline{M}}_1$, $\hat{\underline{M}}_2$, and $\hat{\underline{M}}_3$ and formulas like (65) - (67) yield

$$\underline{E}^b(\hat{\underline{M}}_1) = -\,\underline{u}_\phi\,\frac{jk_b\,e^{-jk_b r}\sin\theta}{2\pi r}\,e^{j\psi} \tag{128a}$$

$$\underline{E}^b(\hat{\underline{M}}_2) = (\underline{u}_\theta\sin\phi + \underline{u}_\phi\cos\theta\cos\phi)\,\frac{jk_b\,e^{-jk_b r}}{2\pi r}\,e^{j\psi} \tag{128b}$$

$$\underline{E}^b(\hat{\underline{M}}_3) = -\,(\underline{u}_\theta\cos\theta\sin\phi + \underline{u}_\phi\cos\phi)\,\frac{k_b^2\,e^{-jk_b r}}{2\pi r}\,e^{j\psi} \tag{128c}$$

where

$$\psi = k_b\sin\theta\,(x'\cos\phi + b\sin\phi) \tag{129}$$

Here, θ is the angle measured from the positive z axis, and $\underline{u}_\theta$ is the unit vector in the θ direction. Also, ϕ is the angle measured in the xy plane from the positive x axis, and $\underline{u}_\phi$ is the unit vector in the ϕ direction. Finally, r is the distance from the origin. The factor $e^{j\psi}$ in (128) is due to the fact that the aperture is located not at the origin but at (x=x', y=b, z=0).

In the waveguide region, the far field due to the aperture is given by (111) and (112) where E_r and E_t are given by (115) and (118), respectively. The V's in (115) and (118) are given by (126). In the half-space region ($y > b$), the far field due to the aperture is $\underline{E}^b(\hat{\underline{M}})$ given by (60) which is

$$\underline{E}^b(\hat{\underline{M}}) = \sum_{i=1}^{3} V_i \underline{E}^b(\hat{\underline{M}}_i) \tag{130}$$

In (130), the V's are given by (126), and $\underline{E}^b(\hat{\underline{M}}_i)$, $i = 1,2,3$, is given by (128).

APPENDIX A. CONSERVATION OF POWER

In Appendix A, it is shown that the moment solution for the current elements $\underline{K\ell}$ and $\underline{I\ell}$ in Fig. 4 satisfies conservation of power. Conservation of power means that the power entering the aperture from region a is equal to the power flowing from the aperture into region b. Applied to the situation in Fig. 4, conservation of power states that the power $-P^a$ absorbed by the current elements $-\underline{K\ell}$ and $-\underline{I\ell}$ on the left-hand side of the complete screen in Fig. 4 should be equal to the power P^b supplied by the current elements $\underline{K\ell}$ and $(\varepsilon_b/\varepsilon_a)\ \underline{I\ell}$ on the right-hand side of the complete screen in Fig. 4. Note that the powers P^a and P^b used here in Appendix A are different from those used in the main text. In the main text, P^a and P^b are calculated with the impressed sources $(\underline{J}^{ia}, \underline{M}^{ia})$ and $(\underline{J}^{ib}, \underline{M}^{ib})$ absent. In Appendix A, P^a and P^b are calculated with the impressed sources present.

The combination of the left-hand current elements $-\underline{K\ell}$ and $-\underline{I\ell}$ is equal to $-\hat{\underline{M}}$ where $\hat{\underline{M}}$ is given by (29) in which $\hat{\underline{M}}_1$ and $\hat{\underline{M}}_2$ are unit magnetic current elements and $\hat{\underline{M}}_3$ is the electric current element $\underline{I\ell} = -j\omega\varepsilon_a$ in the $\underline{n}$ direction. Hence, the power P^a supplied by the combination of $-\underline{K\ell}$ and $-\underline{I\ell}$ is given by

$$P^a = \mathrm{Re}\langle V_1^*\hat{\underline{M}}_1 + V_2^*\hat{\underline{M}}_2,\ \underline{H}^a(-\hat{\underline{M}}) + \underline{H}^{ia}\rangle - \mathrm{Re}\langle V_3^*\hat{\underline{M}}_3,\ \underline{E}^a(-\hat{\underline{M}}) + \underline{E}^{ia}\rangle \tag{A-1}$$

Thanks to (49) and (53), (A-1) becomes

$$P^a = \tilde{\vec{V}}^*\hat{Y}^a\vec{V} + \mathrm{Re}\langle V_1^*\hat{\underline{M}}_1 + V_2^*\hat{\underline{M}}_2,\ \underline{H}^{ia}\rangle - \mathrm{Re}\langle V_3^*\hat{\underline{M}}_3,\ \underline{E}^{ia}\rangle \tag{A-2}$$

By virtue of (44), the symmetry of $\mathrm{Im}[Y^a]$, and the impulsive nature of $\hat{\underline{M}}_1$, $\hat{\underline{M}}_2$, and $\hat{\underline{M}}_3$, (A-2) reduces to

$$P^a = \mathrm{Re}(\tilde{\vec{V}}^* Y^a\vec{V} + V_1^*H_1^{ia} + V_2^*H_2^{ia} + j\omega\varepsilon_a\ V_3^*E_3^{ia}) \tag{A-3}$$

The combination of current elements $\underline{K\ell}$ and $(\varepsilon_b/\varepsilon_a)$ $\underline{I\ell}$ attached to the right-hand side of the complete screen in Fig. 4 is $\underline{\hat{M}}$ given by (29) in which $\underline{\hat{M}}_1$ and $\underline{\hat{M}}_2$ are unit magnetic current elements and $\underline{\hat{M}}_3$ is the electric current element $I\ell = -j\omega\varepsilon_b$ in the $\underline{n}$ direction. In a development similar to (A-1) - (A-3), we obtain

$$P^b = \mathrm{Re}(\tilde{\vec{V}}^* Y^b \vec{V} - V_1^* H_1^{ib} - V_2^* H_2^{ib} - j\omega\varepsilon_b V_3^* E_3^{ib}) \tag{A-4}$$

The sum of (A-3) and (A-4) is

$$P^a + P^b = \mathrm{Re}(\tilde{\vec{V}}^* [[Y^a + Y^b]\vec{V} - \vec{I}^i]) \tag{A-5}$$

where $\vec{I}^i$ is the column vector whose elements are given by (18).

It is evident from the moment equation (5) that the right-hand side of (A-5) vanishes so that

$$-P^a = P^b \tag{A-6}$$

Therefore, the moment solution for the current elements $\underline{K\ell}$ and $\underline{I\ell}$ in Fig. 4 satisfies conservation of power. Conservation of power depends upon accurate evaluation of $\vec{I}^i$ and $\mathrm{Re}[Y^a + Y^b]$. It is interesting to note that, as long as $\mathrm{Im}[Y^a + Y^b]$ is symmetric, $\mathrm{Im}[Y^a + Y^b]$ has no effect on the right-hand side of (A-5) and therefore no influence as to whether power is conserved.

APPENDIX B. RECIPROCITY

In Appendix B, it is shown that the moment solution (3) for the magnetic current $\underline{M}$ satisfies reciprocity. Reciprocity states that

$$\iint (\underline{E}^{ab} \cdot \underline{J}^{ia} - \underline{H}^{ab} \cdot \underline{M}^{ia}) ds = \iint (\underline{E}^{ba} \cdot \underline{J}^{ib} - \underline{H}^{ba} \cdot \underline{M}^{ib}) ds \tag{B-1}$$

where, with reference to the original problem shown in Fig. 1, $(\underline{E}^{ab}, \underline{H}^{ab})$ is the field in region a due to $(\underline{J}^{ib}, \underline{M}^{ib})$, and $(\underline{E}^{ba}, \underline{H}^{ba})$ is the field in region b due to $(\underline{J}^{ia}, \underline{M}^{ia})$. Otherwise stated, $(\underline{E}^{ab}, \underline{H}^{ab})$ would be the field in region a in Fig. 1 if $(\underline{J}^{ia}, \underline{M}^{ia})$ were absent, and $(\underline{E}^{ba}, \underline{H}^{ba})$ would be the field in region b in Fig. 1 if $(\underline{J}^{ib}, \underline{M}^{ib})$ were absent.

It is evident from Figs. 2 and 3 that

$$(\underline{E}^{ab}, \underline{H}^{ab}) = (\underline{E}^{a}(-\underline{M}^{b}), \underline{H}^{a}(-\underline{M}^{b})) \tag{B-2}$$

$$(\underline{E}^{ba}, \underline{H}^{ba}) = (\underline{E}^{b}(\underline{M}^{a}), \underline{H}^{b}(\underline{M}^{a})) \tag{B-3}$$

where $\underline{M}^a$ is the part of $\underline{M}$ due to $(\underline{J}^{ia}, \underline{M}^{ia})$, and $\underline{M}^b$ is the part of $\underline{M}$ due to $(\underline{J}^{ib}, \underline{M}^{ib})$. Otherwise stated, $\underline{M}^a$ is the value of $\underline{M}$ which would result if $(\underline{J}^{ib}, \underline{M}^{ib})$ were absent, and $\underline{M}^b$ is the value of $\underline{M}$ which would result if $(\underline{J}^{ia}, \underline{M}^{ia})$ were absent. Substitution of (B-2) and (B-3) into (B-1) gives

$$-\iint (\underline{E}^{a}(\underline{M}^{b}) \cdot \underline{J}^{ia} - \underline{H}^{a}(\underline{M}^{b}) \cdot \underline{M}^{ia}) ds = \iint (\underline{E}^{b}(\underline{M}^{a}) \cdot \underline{J}^{ib} - \underline{H}^{b}(\underline{M}^{a}) \cdot \underline{M}^{ib}) ds \tag{B-4}$$

Equation (B-4) is equivalent to (B-1). Hence, the moment solution (3) for the magnetic current $\underline{M}$ will satisfy reciprocity if the moment solutions (3) for $\underline{M}^a$ and $\underline{M}^b$ satisfy (B-4).

The moment solutions (3) for $\underline{M}^a$ and $\underline{M}^b$ are given by

$$\underline{M}^a = V_1^a \underline{M}_1 + V_2^a \underline{M}_2 + V_3^a \underline{M}_3 \tag{B-5}$$

$$\underline{M}^b = V_1^b \underline{M}_1 + V_2^b \underline{M}_2 + V_3^b \underline{M}_3 \tag{B-6}$$

where V_1^a, V_2^a, and V_3^a are the elements of the column vector $\vec{V}^a$ which satisfies

$$[Y^a + Y^b]\vec{V}^a = \vec{I}^{ia} \tag{B-7}$$

and V_1^b, V_2^b, and V_3^b are the elements of the column vector $\vec{V}^b$ which satisfies

$$[Y^a + Y^b]\vec{V}^b = \vec{I}^{ib} \tag{B-8}$$

In (B-7) and (B-8), $\vec{I}^{ia}$ and $\vec{I}^{ib}$ are column vectors whose elements are, with regard to (18), given by

$$\begin{aligned} I_1^{ia} &= - H_1^{ia} \\ I_2^{ia} &= - H_2^{ia} \\ I_3^{ia} &= - j\omega\varepsilon_a E_3^{ia} \end{aligned} \tag{B-9}$$

and

$$\begin{aligned} I_1^{ib} &= H_1^{ib} \\ I_2^{ib} &= H_2^{ib} \\ I_3^{ib} &= j\omega\varepsilon_b E_3^{ib} \end{aligned} \tag{B-10}$$

If it is assumed that $\underline{M}^a$ and $\underline{M}^b$ are given by (B-5) and (B-6) and that $(\underline{J}^{ia}, \underline{M}^{ia})$ and $(\underline{J}^{ib}, \underline{M}^{ib})$ are remote from the aperture, then (B-4) is equivalent to

$$-\iint(\underline{E}^a(\hat{\underline{M}}^b) \cdot \underline{J}^{ia} - \underline{H}^a(\hat{\underline{M}}^b) \cdot \underline{M}^{ia})ds = \iint(\underline{E}^b(\hat{\underline{M}}^a) \cdot \underline{J}^{ib} - \underline{H}^b(\hat{\underline{M}}^a) \cdot \underline{M}^{ib})ds \tag{B-11}$$

where

$$\underline{\hat{M}}^a = V_1^a\underline{\hat{M}}_1 + V_2^a\underline{\hat{M}}_2 + V_3^a\underline{\hat{M}}_3 \tag{B-12}$$

$$\underline{\hat{M}}^b = V_1^b\underline{\hat{M}}_1 + V_2^b\underline{\hat{M}}_2 + V_3^b\underline{\hat{M}}_3 \tag{B-13}$$

Here, $\underline{\hat{M}}_1$ and $\underline{\hat{M}}_2$ are the unit magnetic current elements $K\ell = 1$ in the $\underline{t}_1$ and $\underline{t}_2$ directions, respectively. As used in (B-11), the $\underline{\hat{M}}_3$ which appears in (B-12) is the electric current element $I\ell = -j\omega\varepsilon_b$ in the $\underline{n}$ direction. As used in (B-11), the $\underline{\hat{M}}_3$ which appears in (B-13) is the electric current element $I\ell = -j\omega\varepsilon_a$ in the $\underline{n}$ direction.

Applying reciprocity first to the sources $\underline{\hat{M}}^b$ and $(\underline{J}^{ia}, \underline{M}^{ia})$ to the left of the complete screen and then to the sources $\underline{\hat{M}}^a$ and $(\underline{J}^{ib}, \underline{M}^{ib})$ to the right of the complete screen, we find that (B-11) is equivalent to

$$V_1^b H_1^{ia} + V_2^b H_2^{ia} + j\omega\varepsilon_a V_3^b E_3^{ia} = - V_1^a H_1^{ib} - V_2^a H_2^{ib} - j\omega\varepsilon_b V_3^a E_3^{ib} \tag{B-14}$$

Equation (B-14) is always equivalent to (B-11) regardless of the values of the V's.

It will now be shown that the V's which satisfy (B-7) and (B-8) also satisfy (B-14). In matrix form, (B-14) is

$$-\tilde{V}^b\vec{I}^{ia} = - \tilde{V}^a\vec{I}^{ib} \tag{B-15}$$

where $\tilde{V}^a$ and $\tilde{V}^b$ are the transposes of the column vectors $\vec{V}^a$ and $\vec{V}^b$ in (B-7) and (B-8). The column vectors $\vec{I}^{ia}$ and $\vec{I}^{ib}$ in (B-15) are the same as those in (B-7) and (B-8). Multiplication of (B-7) by $\tilde{V}^b$ from the left gives

$$\tilde{V}^b[Y^a + Y^b]\vec{V}^a = \tilde{V}^b\vec{I}^{ia} \tag{B-16}$$

Multiplication of (B-8) by $\tilde{V}^a$ from the left gives

$$\tilde{V}^a[Y^a + Y^b]\vec{V}^b = \tilde{V}^a\vec{I}^{ib} \tag{B-17}$$

Because $[Y^a + Y^b]$ is a symmetric matrix, the left-hand sides of (B-16) and (B-17) are equal to each other. Therefore, the right-hand sides of (B-16) and (B-17) are equal to each other so that (B-15) is true.

In the previous paragraph, we found that the column vectors $\vec{V}^a$ and $\vec{V}^b$ of the V's in the moment solutions (B-5) and (B-6) satisfy (B-15). Backtracking, we see that these V's satisfy (B-14) which is equivalent to (B-11). Since (B-11) is equivalent to (B-4), we conclude that the moment solutions for $\underline{M}^a$ and $\underline{M}^b$ satisfy the statement (B-4) of reciprocity. In other words, the moment solution (3) for the magnetic current $\underline{M}$ satisfies reciprocity.

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